Chapter XVIII

Review of classical electrodynamics

Introduction

In the three previous chapters, we studied ensembles of identical particles, which allowed us to introduce the concept of quantum field operators. We now begin a new series of three chapters where this quantum field concept is applied to an important particular case: the electromagnetic field, made of identical bosons called "photons". We start by noting that, in classical electromagnetism, the dynamics of the different field modes is exactly similar to that of a series of harmonic oscillators. Each of these modes may be quantized by the same method as that used for an elementary harmonic oscillator, for a single particle; this method has the great advantage of simplicity. It requires, however, establishing beforehand the equivalence between modes of the classical electrodynamic field and harmonic oscillators; this is the main purpose of the present chapter.

For the presentation to be self-contained, we first review a certain number of properties of classical electromagnetism. One complement is also devoted to a synthetic

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presentation of the Lagrangian formalism applied to this case. The reader already familiar with those aspects of classical electrodynamics may wish to go directly to the quantum treatment presented in Chapter XIX.

We start in § A with the equations of Maxwell-Lorentz describing the coupled evolution of the electric field $\boldsymbol{E}(\boldsymbol{r}, t)$, the magnetic field $\boldsymbol{B}(\boldsymbol{r}, t)$ and the coordinates and speeds of the particles acting as source for this electromagnetic field[1]. We shall give the expressions for a certain number of constants of motion, such as the energy, or the linear and angular momenta of the global system "field + particles". The vector potential $\boldsymbol{A}(\boldsymbol{r}, t)$ and scalar potential $U(\boldsymbol{r}, t)$ will also be introduced, as well as the gauge transformations that can be performed on these potentials.

We shall then show that it is useful to take the spatial Fourier transforms of these fields, since in the reciprocal space, Maxwell's equations have a simpler form. For a free electromagnetic field (in the absence of charged particles), they are no longer partial differential equations, as in ordinary space, but ordinary time-dependent differential equations. Furthermore, the concept of longitudinal or transverse field vectors has a clear geometrical significance in the reciprocal space[2]. A field vector $\tilde{\boldsymbol{V}}(\boldsymbol{k}, t)$ is longitudinal if $\tilde{\boldsymbol{V}}(\boldsymbol{k}, t)$ is parallel to $\boldsymbol{k}$ at every point $\boldsymbol{k}$ of the reciprocal space, transverse if $\tilde{\boldsymbol{V}}(\boldsymbol{k}, t)$ is perpendicular to $\boldsymbol{k}$ at every point $\boldsymbol{k}$. We will show that two of the four Maxwell's equations yield the value of the longitudinal electrical and magnetic fields, whereas the other two describe the evolution of the transverse fields. It will become clear that the longitudinal electric field is simply the Coulomb electrostatic field created by the charged particles. Consequently, it is not an independent field variable since it only depends on the coordinates of the particles[3]. Furthermore, choosing the Coulomb gauge amounts to choosing the longitudinal potential vector equal to zero; this permits eliminating the longitudinal fields from the expressions for all the physical quantities.

In § B, we establish the equivalence between the radiation field and an ensemble of one-dimensional harmonic oscillators. Maxwell's equations for transverse fields enable introducing linear combinations of the vector potentials and transverse electric fields, whose time evolution, in the absence of particles, is of the form $e^{-i\omega t}$ where $\omega = ck$. These variables, called *normal variables*, thus describe the eigenmodes of the free field vibrations. The dynamics of each of these eigenmodes is similar to that of a one-dimensional harmonic oscillator. The normal mode variable is the equivalent of the linear combination of the position and velocity of the associated operator, and becomes, in the quantization process, the annihilation operator, fundamental in the quantum theory of the harmonic oscillator. Replacing the normal variables and their complex conjugates by annihilation and creation operators will yield, in Chapter XIX, the expressions for the various operators of the quantum theory.

[1] We assume that the speeds of the particles are small compared to the speed of light, so as to use a non-relativistic description.

[2] We shall note $\tilde{G}(\boldsymbol{k})$ the spatial Fourier transform of $G(\boldsymbol{r})$, the symbol "tilde" allowing a clear distinction between the functions in ordinary and reciprocal space.

[3] As for the longitudinal magnetic field, it is simply zero.

A. Classical electrodynamics

A-1. Basic equations and relations

A-1-a. Maxwell's equations

There are four Maxwell's equations in vacuum, and in the presence of sources:

$$\boldsymbol{\nabla} \cdot \boldsymbol{E}(\boldsymbol{r}, t) = \frac{1}{\varepsilon_0}\rho(\boldsymbol{r}, t) \tag{A-1a}$$

$$\boldsymbol{\nabla} \cdot \boldsymbol{B}(\boldsymbol{r}, t) = 0 \tag{A-1b}$$

$$\boldsymbol{\nabla} \times \boldsymbol{E}(\boldsymbol{r}, t) = -\frac{\partial}{\partial t}\boldsymbol{B}(\boldsymbol{r}, t) \tag{A-1c}$$

$$\boldsymbol{\nabla} \times \boldsymbol{B}(\boldsymbol{r}, t) = \frac{1}{c^2}\frac{\partial}{\partial t}\boldsymbol{E}(\boldsymbol{r}, t) + \frac{1}{\varepsilon_0 c^2}\boldsymbol{j}(\boldsymbol{r}, t) \tag{A-1d}$$

where c is the velocity of light in vacuum and ε_0 the vacuum permittivity. These equations yield the divergence and the curl of the electric field $\boldsymbol{E}(\boldsymbol{r}, t)$ and the magnetic field $\boldsymbol{B}(\boldsymbol{r}, t)$. The charge density $\rho(\boldsymbol{r}, t)$ and current density $\boldsymbol{j}(\boldsymbol{r}, t)$ appearing in those equations can be expressed, in the non-relativistic limit, in terms of the positions $\boldsymbol{r}_a(t)$ and the speeds $\boldsymbol{v}_a(t) = \mathrm{d}\boldsymbol{r}_a(t)/\mathrm{d}t$ of the various particles a of the system, each having a mass m_a and a charge q_a:

$$\rho(\boldsymbol{r}, t) = \sum_a q_a \delta\left[\boldsymbol{r} - \boldsymbol{r}_a(t)\right] \tag{A-2a}$$

$$\boldsymbol{j}(\boldsymbol{r}, t) = \sum_a q_a \boldsymbol{v}_a(t) \delta\left[\boldsymbol{r} - \boldsymbol{r}_a(t)\right] \tag{A-2b}$$

A-1-b. Lorentz Equations

Lorentz equations describe the dynamics of each particle a submitted to the electric and magnetic forces exerted by the fields $\boldsymbol{E}(\boldsymbol{r}, t)$ and $\boldsymbol{B}(\boldsymbol{r}, t)$:

$$m_a \frac{\mathrm{d}^2}{\mathrm{d}t^2}\boldsymbol{r}_a(t) = q_a\left[\boldsymbol{E}\left(\boldsymbol{r}_a(t), t\right) + \boldsymbol{v}_a(t) \times \boldsymbol{B}\left(\boldsymbol{r}_a(t), t\right)\right] \tag{A-3}$$

The particle and field evolutions are coupled: the particles move under the effect of the forces the fields exert on them, but they also act as sources for the evolution of those fields.

A-1-c. Constants of motion

Definitions (A-2a) of $\rho(\boldsymbol{r}, t)$ and (A-2b) of $\boldsymbol{j}(\boldsymbol{r}, t)$ lead to the continuity equation:

$$\frac{\partial}{\partial t}\rho(\boldsymbol{r}, t) + \boldsymbol{\nabla} \cdot \boldsymbol{j}(\boldsymbol{r}, t) = 0 \tag{A-4}$$

which implies the time invariance of the total charge of the particle system:

$$Q = \int \mathrm{d}^3 r \rho(\boldsymbol{r}, t) = \sum_a q_a \tag{A-5}$$

Other constants of motion exist: the total energy H, the total momentum $\boldsymbol{P}$ and the total angular momentum $\boldsymbol{J}$ of the system field + particles. They are respectively given by:

$$H = \sum_a \frac{1}{2} m_a \boldsymbol{v}_a^2(t) + \frac{\varepsilon_0}{2} \int \mathrm{d}^3r \left[\boldsymbol{E}^2(\boldsymbol{r},t) + c^2 \boldsymbol{B}^2(\boldsymbol{r},t) \right] \tag{A-6a}$$

$$\boldsymbol{P} = \sum_a m_a \boldsymbol{v}_a(t) + \varepsilon_0 \int \mathrm{d}^3r \; \boldsymbol{E}(\boldsymbol{r},t) \times \boldsymbol{B}(\boldsymbol{r},t) \tag{A-6b}$$

$$\boldsymbol{J} = \sum_a \boldsymbol{r}_a(t) \times m_a \boldsymbol{v}_a(t) + \varepsilon_0 \int \mathrm{d}^3r \; \boldsymbol{r} \times \left[\boldsymbol{E}(\boldsymbol{r},t) \times \boldsymbol{B}(\boldsymbol{r},t) \right] \tag{A-6c}$$

Using (A-1) and (A-3), we can verify that the derivatives with respect to time of H, $\boldsymbol{P}$ and $\boldsymbol{J}$ are indeed zero (for H and $\boldsymbol{P}$, see for example exercise 1 in Complement $\mathrm{C_I}$ of [16] and its correction).

A-1-d. Scalar and vector potentials: gauge transformations

As we already saw in Complement $\mathrm{H_{III}}$, the fields $\boldsymbol{E}(\boldsymbol{r},t)$ and $\boldsymbol{B}(\boldsymbol{r},t)$ can always be written in the form:

$$\boldsymbol{E}(\boldsymbol{r},t) = -\boldsymbol{\nabla} U(\boldsymbol{r},t) - \frac{\partial}{\partial t} \boldsymbol{A}(\boldsymbol{r},t) \tag{A-7a}$$

$$\boldsymbol{B}(\boldsymbol{r},t) = \boldsymbol{\nabla} \times \boldsymbol{A}(\boldsymbol{r},t) \tag{A-7b}$$

where $\boldsymbol{A}(\boldsymbol{r},t)$ and $U(\boldsymbol{r},t)$ are the vector and scalar potentials defining a *gauge*. For any function $\chi(\boldsymbol{r},t)$ of $\boldsymbol{r}$ and of t, the transformation of these potentials obeying the relations:

$$\boldsymbol{A}(\boldsymbol{r},t) \Rightarrow \boldsymbol{A}'(\boldsymbol{r},t) = \boldsymbol{A}(\boldsymbol{r},t) + \boldsymbol{\nabla}\chi(\boldsymbol{r},t) \tag{A-8a}$$

$$U(\boldsymbol{r},t) \Rightarrow U'(\boldsymbol{r},t) = U(\boldsymbol{r},t) - \frac{\partial}{\partial t}\chi(\boldsymbol{r},t) \tag{A-8b}$$

leads to the same expression for $\boldsymbol{E}(\boldsymbol{r},t)$ and $\boldsymbol{B}(\boldsymbol{r},t)$; the same physical fields can therefore be represented by several different potentials $\boldsymbol{A}(\boldsymbol{r},t)$ and $U(\boldsymbol{r},t)$. The transformation (A-8) associated with the function $\chi(\boldsymbol{r},t)$ is called a *gauge transformation.*

Relations (A-8) allow a flexibility on the choice of the gauge $\{\boldsymbol{A}, U\}$, which allows introducing an additional condition. The Coulomb gauge, which we will use in this chapter and the following, is defined by the condition:

$$\boldsymbol{\nabla} \cdot \boldsymbol{A}(\boldsymbol{r},t) = 0 \tag{A-9}$$

A geometrical interpretation of condition (A-9) in the reciprocal space will be given later.

A-2. Description in the reciprocal space

Using Fourier transforms, the equations of electrodynamics can be put in a form that simplifies calculations.

A-2-a. Spatial Fourier transforms

Let us introduce the Fourier transform of the electric field $\boldsymbol{E}(\boldsymbol{r},t)$:

$$\tilde{\boldsymbol{E}}(\boldsymbol{k},t) = \frac{1}{(2\pi)^{3/2}} \int \mathrm{d}^3 r\, \boldsymbol{E}(\boldsymbol{r},t) e^{-i\boldsymbol{k}\cdot\boldsymbol{r}} \tag{A-10}$$

which enables us to write $\boldsymbol{E}(\boldsymbol{r},t)$ as:

$$\boldsymbol{E}(\boldsymbol{r},t) = \frac{1}{(2\pi)^{3/2}} \int \mathrm{d}^3 k\, \tilde{\boldsymbol{E}}(\boldsymbol{k},t) e^{i\boldsymbol{k}\cdot\boldsymbol{r}} \tag{A-11}$$

Analogous expressions can be written for all the physical quantities we just introduced: magnetic field, charge and current densities, scalar and vector potentials.

It will be useful in what follows to recall the Parseval-Plancherel relation (Appendix I, § 2-c) showing the identity of the scalar products of two functions expressed in position space or in reciprocal space[4]:

$$\int \mathrm{d}^3 r\, F^*(\boldsymbol{r}) G(\boldsymbol{r}) = \int \mathrm{d}^3 k\, \tilde{F}^*(\boldsymbol{k}) \tilde{G}(\boldsymbol{k}) \tag{A-12}$$

and the fact that the product of two functions in reciprocal space, is the Fourier transform of their convolution in position space:

$$\tilde{F}(\boldsymbol{k})\tilde{G}(\boldsymbol{k}) \underset{\mathrm{FT}}{\Leftrightarrow} \frac{1}{(2\pi)^{3/2}} \int \mathrm{d}^3 r'\, F(\boldsymbol{r}') G(\boldsymbol{r}-\boldsymbol{r}') \tag{A-13}$$

A-2-b. Maxwell's equations in reciprocal space

Maxwell's equations take on a simpler form in the reciprocal space, clearly showing the differences between the longitudinal and transverse components of the various fields. Any vector field $\tilde{\boldsymbol{V}}(\boldsymbol{k},t)$ can be decomposed into a longitudinal field $\tilde{\boldsymbol{V}}_{\|}(\boldsymbol{k},t)$, parallel at any point $\boldsymbol{k}$ to the vector $\boldsymbol{k}$, and a transverse field $\tilde{\boldsymbol{V}}_{\perp}(\boldsymbol{k},t)$ perpendicular to $\boldsymbol{k}$:

$$\tilde{\boldsymbol{V}}(\boldsymbol{k},t) = \tilde{\boldsymbol{V}}_{\|}(\boldsymbol{k},t) + \tilde{\boldsymbol{V}}_{\perp}(\boldsymbol{k},t) \tag{A-14}$$

with:

$$\tilde{\boldsymbol{V}}_{\|}(\boldsymbol{k},t) = \boldsymbol{\kappa}\left(\boldsymbol{\kappa}\cdot\tilde{\boldsymbol{V}}(\boldsymbol{k},t)\right) = \boldsymbol{k}\left(\boldsymbol{k}\cdot\tilde{\boldsymbol{V}}(\boldsymbol{k},t)\right)/k^2 \tag{A-15a}$$

$$\tilde{\boldsymbol{V}}_{\perp}(\boldsymbol{k},t) = \tilde{\boldsymbol{V}}(\boldsymbol{k},t) - \tilde{\boldsymbol{V}}_{\|}(\boldsymbol{k},t) \tag{A-15b}$$

where

$$\boldsymbol{\kappa} = \boldsymbol{k}/k \tag{A-16}$$

is the unit vector along $\boldsymbol{k}$.

[4]The space of the vectors $\boldsymbol{r}$ (ordinary space) is called "position space" whereas "reciprocal space" is the space of the wave vectors $\boldsymbol{k}$.

As the operator $\boldsymbol{\nabla}$ in position space corresponds to the operator $i\boldsymbol{k}$ in reciprocal space, Maxwell's equations (A-1) become in reciprocal space:

$$i\boldsymbol{k}\cdot\tilde{\boldsymbol{E}}(\boldsymbol{k},t)=\frac{1}{\varepsilon_0}\tilde{\rho}(\boldsymbol{k},t) \tag{A-17a}$$

$$i\boldsymbol{k}\cdot\tilde{\boldsymbol{B}}(\boldsymbol{k},t)=0 \tag{A-17b}$$

$$i\boldsymbol{k}\times\tilde{\boldsymbol{E}}(\boldsymbol{k},t)=-\frac{\partial}{\partial t}\tilde{\boldsymbol{B}}(\boldsymbol{k},t) \tag{A-17c}$$

$$i\boldsymbol{k}\times\tilde{\boldsymbol{B}}(\boldsymbol{k},t)=\frac{1}{c^2}\frac{\partial}{\partial t}\tilde{\boldsymbol{E}}(\boldsymbol{k},t)+\frac{1}{\varepsilon_0 c^2}\tilde{\boldsymbol{j}}(\boldsymbol{k},t) \tag{A-17d}$$

Taking into account definitions (A-15) for the longitudinal and transverse components of a vector field, the first two equations (A-17a) and (A-17b) determine the longitudinal parts, projections of the fields $\tilde{\boldsymbol{E}}(\boldsymbol{k},t)$ and $\tilde{\boldsymbol{B}}(\boldsymbol{k},t)$ onto $\boldsymbol{k}$:

$$\tilde{\boldsymbol{E}}_{\parallel}(\boldsymbol{k},t)=-\frac{i}{\varepsilon_0}\tilde{\rho}(\boldsymbol{k},t)\frac{\boldsymbol{k}}{k^2} \tag{A-18a}$$

$$\tilde{\boldsymbol{B}}_{\parallel}(\boldsymbol{k},t)=\boldsymbol{0} \tag{A-18b}$$

The last two equations (A-17c) and (A-17d) yield the rate of change $\partial\tilde{\boldsymbol{E}}(\boldsymbol{k},t)/\partial t$ and $\partial\tilde{\boldsymbol{B}}(\boldsymbol{k},t)/\partial t$ of the fields $\tilde{\boldsymbol{E}}(\boldsymbol{k},t)$ and $\tilde{\boldsymbol{B}}(\boldsymbol{k},t)$, and are the equations of motion of these fields. In the absence of sources ($\tilde{\boldsymbol{j}}(\boldsymbol{k},t)=\boldsymbol{0}$), i.e. for what we will call a "free" field, they are time-dependent differential equations, and no longer partial derivative equations as is the case in position space.

A-2-c. Longitudinal electric and magnetic fields

Equation (A-18b) shows the longitudinal magnetic field $\tilde{\boldsymbol{B}}_{\parallel}(\boldsymbol{k},t)$ is zero. Equation (A-18a) expresses $\tilde{\boldsymbol{E}}_{\parallel}(\boldsymbol{k},t)$ as a product of two functions of $\boldsymbol{k}$, $\tilde{\rho}(\boldsymbol{k},t)$ and $-i\boldsymbol{k}/\varepsilon_0 k^2$ whose Fourier transforms are written (relation (63) of Appendix I):

$$\tilde{\rho}(\boldsymbol{k},t) \underset{\text{FT}}{\leftrightarrow} \rho(\boldsymbol{r},t) \tag{A-19a}$$

$$-\frac{i}{\varepsilon_0}\frac{\boldsymbol{k}}{k^2} \underset{\text{FT}}{\leftrightarrow} \frac{(2\pi)^{3/2}}{4\pi\varepsilon_0}\frac{\boldsymbol{r}}{r^3} \tag{A-19b}$$

Using relation (A-13) then leads to:

$$\begin{aligned}\boldsymbol{E}_{\parallel}(\boldsymbol{r},t)&=\frac{1}{4\pi\varepsilon_0}\int \mathrm{d}^3r'\,\rho(\boldsymbol{r}',t)\frac{\boldsymbol{r}-\boldsymbol{r}'}{|\boldsymbol{r}-\boldsymbol{r}'|^3}\\ &=\frac{1}{4\pi\varepsilon_0}\sum_a q_a\frac{\boldsymbol{r}-\boldsymbol{r}_a(t)}{|\boldsymbol{r}-\boldsymbol{r}_a(t)|^3}\end{aligned} \tag{A-20}$$

This means that at time t, the longitudinal electric field coincides with the Coulomb field produced by the charge distribution $\rho(\boldsymbol{r},t)$, computed as if this distribution were static and fixed at that instant t.

Comment

The fact that the longitudinal electric field instantaneously follows the evolution of the charge distribution $\rho(\boldsymbol{r},t)$ should not lead us to believe in an *action at a distance* propagating at an infinite speed. The contribution of the transverse field must also be taken into account, as only the total electric field $\boldsymbol{E} = \boldsymbol{E}_{\parallel} + \boldsymbol{E}_{\perp}$ has a real physical meaning. It can be shown that the transverse electric field also has an instantaneous component, which balances exactly the longitudinal component so that the total field is always retarded (to $t - |r - r'|/c$), as the electromagnetic interactions propagate at the speed of light c (see exercise 3 and its correction in Complement C_I of reference [16]).

The previous results show that the longitudinal fields are not independent quantities: they are either zero (in the case of the longitudinal magnetic field), or simply related to the particle coordinates $\boldsymbol{r}_a(t)$ (in the case of the longitudinal electric field, whose expression is given by (A-20)).

A-2-d. Time evolution of the transverse fields

Now that we showed that the first two Maxwell's equations determine the longitudinal part of the fields, let us consider the last two equations (A-17c) and (A-17d) and focus on their transverse components. Since $\boldsymbol{k} \times \boldsymbol{E} = \boldsymbol{k} \times \boldsymbol{E}_{\perp}$, they can be rewritten as:

$$\frac{\partial}{\partial t}\tilde{\boldsymbol{B}}(\boldsymbol{k},t) = -i\boldsymbol{k} \times \tilde{\boldsymbol{E}}_{\perp}(\boldsymbol{k},t) \tag{A-21a}$$

$$\frac{\partial}{\partial t}\tilde{\boldsymbol{E}}_{\perp}(\boldsymbol{k},t) = ic^2\boldsymbol{k} \times \tilde{\boldsymbol{B}}(\boldsymbol{k},t) - \frac{1}{\varepsilon_0}\tilde{\boldsymbol{j}}_{\perp}(\boldsymbol{k},t) \tag{A-21b}$$

which yield the time evolution of the transverse fields $\tilde{\boldsymbol{E}}_{\perp}(\boldsymbol{k},t)$ and $\tilde{\boldsymbol{B}}(\boldsymbol{k},t)$.

Comment

One can also study the longitudinal projections of the two Maxwell's equations (A-17c) and (A-17d). The result is trivial for the first one: as both sides of the equation are transverse, their longitudinal projections are zero. As for the second equation, (A-17d), it leads to:

$$\frac{\partial}{\partial t}\tilde{\boldsymbol{E}}_{\parallel}(\boldsymbol{k},t) + \frac{1}{\varepsilon_0}\tilde{\boldsymbol{j}}_{\parallel}(\boldsymbol{k},t) = 0. \tag{A-22}$$

Taking the scalar product of $\boldsymbol{k}$ with each side of this equation, using (A-18a) and the fact that $\boldsymbol{k}\cdot\tilde{\boldsymbol{j}} = \boldsymbol{k}\cdot\tilde{\boldsymbol{j}}_{\parallel}$, we find:

$$\frac{\partial}{\partial t}\tilde{\rho}(\boldsymbol{k},t) + i\boldsymbol{k}\cdot\tilde{\boldsymbol{j}}(\boldsymbol{k},t) = 0 \tag{A-23}$$

which is simply the continuity equation (A-4) in the reciprocal space, and does not provide any new information.

A-2-e. Potentials

In the reciprocal space, relations (A-7a) and (A-7b) between fields and potentials become:

$$\tilde{\boldsymbol{E}}(\boldsymbol{k},t) = -i\boldsymbol{k}\tilde{U}(\boldsymbol{k},t) - \frac{\partial}{\partial t}\tilde{\boldsymbol{A}}(\boldsymbol{k},t) \tag{A-24a}$$

$$\tilde{\boldsymbol{B}}(\boldsymbol{k},t) = i\boldsymbol{k} \times \tilde{\boldsymbol{A}}(\boldsymbol{k},t) \tag{A-24b}$$

and the gauge transformations relations (A-8a) and (A-8b) are written:

$$\tilde{\boldsymbol{A}}(\boldsymbol{k},t) \to \tilde{\boldsymbol{A}}'(\boldsymbol{k},t) = \tilde{\boldsymbol{A}}(\boldsymbol{k},t) + i\boldsymbol{k}\tilde{\chi}(\boldsymbol{k},t) \tag{A-25a}$$

$$\tilde{U}(\boldsymbol{k},t) \to \tilde{U}'(\boldsymbol{k},t) = \tilde{U}(\boldsymbol{k},t) - \frac{\partial}{\partial t}\tilde{\chi}(\boldsymbol{k},t) \tag{A-25b}$$

where $\tilde{\chi}(\boldsymbol{k},t)$ is the Fourier transform of $\chi(\boldsymbol{r},t)$.

Since the last term in (A-25a) is a longitudinal vector, it is clear that a gauge transformation does not change the transverse part $\tilde{\boldsymbol{A}}_\perp(\boldsymbol{k},t)$, which thus defines a gauge invariant physical field:

$$\tilde{\boldsymbol{A}}'_\perp(\boldsymbol{k},t) = \tilde{\boldsymbol{A}}_\perp(\boldsymbol{k},t) \tag{A-26}$$

Since $\boldsymbol{k} \times \tilde{\boldsymbol{A}}_\parallel = \boldsymbol{0}$, the transverse projections of relations (A-24a) and (A-24b) yield the equations:

$$\tilde{\boldsymbol{E}}_\perp(\boldsymbol{k},t) = -\frac{\partial}{\partial t}\tilde{\boldsymbol{A}}_\perp(\boldsymbol{k},t) \tag{A-27a}$$

$$\tilde{\boldsymbol{B}}(\boldsymbol{k},t) = i\boldsymbol{k} \times \tilde{\boldsymbol{A}}_\perp(\boldsymbol{k},t) \tag{A-27b}$$

Note that equation (A-27b) allows expressing $\tilde{\boldsymbol{A}}_\perp(\boldsymbol{k},t)$ as a function of $\tilde{\boldsymbol{B}}(\boldsymbol{k},t)$, as we now show. Taking the vector product of $\boldsymbol{k}$ with each side of this equation, and using the identity:

$$\boldsymbol{a} \times (\boldsymbol{b} \times \boldsymbol{c}) = (\boldsymbol{a}\cdot\boldsymbol{c})\boldsymbol{b} - (\boldsymbol{a}\cdot\boldsymbol{b})\boldsymbol{c}, \tag{A-28}$$

and the fact that $\boldsymbol{k} \cdot \tilde{\boldsymbol{A}}_\perp(\boldsymbol{k},t) = 0$, we get:

$$\tilde{\boldsymbol{A}}_\perp(\boldsymbol{k},t) = \frac{i}{k^2}\left(\boldsymbol{k} \times \tilde{\boldsymbol{B}}(\boldsymbol{k},t)\right) \tag{A-29}$$

This equation, together with equation (A-27a), allow rewriting the two time evolution equations (A-21a) and (A-21b) for the transverse fields in a form only involving $\tilde{\boldsymbol{E}}_\perp(\boldsymbol{k},t)$ and $\tilde{\boldsymbol{A}}_\perp(\boldsymbol{k},t)$:

$$\frac{\partial}{\partial t}\tilde{\boldsymbol{A}}_\perp(\boldsymbol{k},t) = -\tilde{\boldsymbol{E}}_\perp(\boldsymbol{k},t) \tag{A-30a}$$

$$\frac{\partial}{\partial t}\tilde{\boldsymbol{E}}_\perp(\boldsymbol{k},t) = c^2k^2\tilde{\boldsymbol{A}}_\perp(\boldsymbol{k},t) - \frac{1}{\varepsilon_0}\tilde{\boldsymbol{j}}_\perp(\boldsymbol{k},t) \tag{A-30b}$$

In the absence of sources ($\tilde{\boldsymbol{j}}_\perp(\boldsymbol{k},t) = \boldsymbol{0}$), we get two coupled time evolution equations for the transverse fields $\tilde{\boldsymbol{E}}_\perp(\boldsymbol{k},t)$ and $\tilde{\boldsymbol{A}}_\perp(\boldsymbol{k},t)$. They will be useful later on for introducing the field normal variables, and for the demonstration of the equivalence between the transverse field and an ensemble of harmonic oscillators.

Time evolution equation for the transverse potential vector

The time evolution equation for $\tilde{\boldsymbol{A}}_\perp$ can be obtained by replacing $\tilde{\boldsymbol{E}}_\perp$ in (A-30b) by $-\partial\tilde{\boldsymbol{A}}_\perp/\partial t$. We obtain:

$$\left[\frac{\partial^2}{\partial t^2} + c^2k^2\right]\tilde{\boldsymbol{A}}_\perp(\boldsymbol{k},t) = \frac{1}{\varepsilon_0}\tilde{\boldsymbol{j}}_\perp(\boldsymbol{k},t) \tag{A-31}$$

which is written, in the position space:

$$\left[\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \Delta\right]\boldsymbol{A}_\perp(\boldsymbol{r},t) = \frac{1}{\varepsilon_0 c^2}\boldsymbol{j}_\perp(\boldsymbol{r},t) \tag{A-32}$$

A-2-f. Coulomb gauge

Condition $\boldsymbol{\nabla} \cdot \boldsymbol{A}(\boldsymbol{r}, t) = 0$, which defined in (A-9) the Coulomb gauge, becomes in the reciprocal space:

$$i\boldsymbol{k} \cdot \tilde{\boldsymbol{A}}(\boldsymbol{k}, t) = 0 \quad \longleftrightarrow \quad \tilde{\boldsymbol{A}}_{\parallel}(\boldsymbol{k}, t) = 0 \tag{A-33}$$

In the Coulomb gauge, the longitudinal vector potential is therefore equal to zero; there only remains the transverse vector potential, which, as mentioned above, is a physical field.

What can be said about the scalar potential U in the Coulomb gauge? Let us consider the longitudinal part of each side of equation (A-24a). As the last term on the right-hand side is transverse in the Coulomb gauge, we get $\tilde{\boldsymbol{E}}_{\parallel}(\boldsymbol{k}, t) = -i\boldsymbol{k}\tilde{U}(\boldsymbol{k}, t)$, which reads, in position space, $\boldsymbol{E}_{\parallel}(\boldsymbol{r}, t) = -\boldsymbol{\nabla} U(\boldsymbol{r}, t)$. The scalar potential is the potential whose gradient yields the longitudinal electric field. Equation (A-20) then shows that, to within a constant, $U(\boldsymbol{r}, t)$ is equal to:

$$U(\boldsymbol{r}, t) = \frac{1}{4\pi\varepsilon_0} \sum_a q_a \frac{1}{|\boldsymbol{r} - \boldsymbol{r}_a(t)|} \tag{A-34}$$

which is the Coulomb potential created by the charge distribution.

Lorenz gauge

In the present chapter and the next one, we shall mainly use the Coulomb gauge. Another gauge often used, in particular in the clearly covariant formulations of electrodynamics, is the *Lorentz gauge*[5] defined by the condition:

$$\boldsymbol{\nabla} \cdot \boldsymbol{A}(\boldsymbol{r}, t) + \frac{1}{c^2} \frac{\partial}{\partial t} U(\boldsymbol{r}, t) = 0 \tag{A-35}$$

which can be written, using covariant notation:

$$\sum_\mu \partial_\mu A^\mu = 0 \tag{A-36}$$

The condition defining the Lorenz gauge thus keeps the same form in every Lorentz reference frame, which is not the case for the Coulomb gauge (since in relativity, a transverse field of zero divergence in one reference frame is no longer necessarily transverse in another frame). Nevertheless, an advantage of the Coulomb gauge is that it allows the immediate identification, in a given reference frame, of the field variables that are really independent.

A-3. Elimination of the longitudinal fields from the expression of the physical quantities

It will be useful for the following discussion to eliminate the longitudinal fields from the expressions of the total energy H and the total momentum given by equations (A-6a) and (A-6b). We shall express these physical quantities only in terms of the truly independent variables, such as particle coordinates and speeds, and transverse fields.

[5] The danish physicist Ludwig Lorenz is often confused with the dutch physicist Hendrik Lorentz.

A-3-a. Total energy

We start by eliminating the longitudinal electric field from the last term in expression (A-6a). Using the Parseval-Plancherel equality (A-12) and the fact that $\tilde{\boldsymbol{E}}_{\parallel}^*(\boldsymbol{k},t)\cdot\tilde{\boldsymbol{E}}_{\perp}(\boldsymbol{k},t)=0$, we can rewrite this term as:

$$\frac{\varepsilon_0}{2}\int \mathrm{d}^3r\,\left[\boldsymbol{E}^2(\boldsymbol{r},t)+c^2\boldsymbol{B}^2(\boldsymbol{r},t)\right]=H_{\mathrm{long}}+H_{\mathrm{trans}} \tag{A-37}$$

where:

$$H_{\mathrm{long}}=\frac{\varepsilon_0}{2}\int \mathrm{d}^3k\;\tilde{\boldsymbol{E}}_{\parallel}^*(\boldsymbol{k},t)\cdot\tilde{\boldsymbol{E}}_{\parallel}(\boldsymbol{k},t) \tag{A-38a}$$

$$H_{\mathrm{trans}}=\frac{\varepsilon_0}{2}\int \mathrm{d}^3k\;\left[\tilde{\boldsymbol{E}}_{\perp}^*(\boldsymbol{k},t)\cdot\tilde{\boldsymbol{E}}_{\perp}(\boldsymbol{k},t)+c^2\tilde{\boldsymbol{B}}^*(\boldsymbol{k},t)\cdot\tilde{\boldsymbol{B}}(\boldsymbol{k},t)\right] \tag{A-38b}$$

In (A-38a), we replace $\tilde{\boldsymbol{E}}_{\parallel}(\boldsymbol{k},t)$ by expression (A-18a). We get, taking (A-12) and (A-13) into account:

$$\begin{aligned}H_{\mathrm{long}}&=\frac{1}{2\varepsilon_0}\int \mathrm{d}^3k\;\frac{\tilde{\rho}^*(\boldsymbol{k},t)\tilde{\rho}(\boldsymbol{k},t)}{k^2}\\&=\frac{1}{8\pi\varepsilon_0}\int\int \mathrm{d}^3r\,\mathrm{d}^3r'\;\frac{\tilde{\rho}(\boldsymbol{r},t)\tilde{\rho}(\boldsymbol{r}',t)}{|\boldsymbol{r}-\boldsymbol{r}'|}\\&=\sum_a h_{\mathrm{Coul}}^a+\frac{1}{8\pi\epsilon_0}\sum_{a\neq b}\frac{q_aq_b}{|\boldsymbol{r}_a-\boldsymbol{r}_b|}=V_{\mathrm{Coul}}\end{aligned} \tag{A-39}$$

The longitudinal field energy is thus equal to the Coulomb electrostatic energy V_{Coul} of the charge distribution $\rho(\boldsymbol{r},t)$. In addition to the Coulomb interaction energy between different particles a and b, V_{Coul} also contains the energy h_{Coul}^a of the Coulomb field of each particle a, which diverges for point particles.

Expression (A-38b) for H_{trans} can be rewritten as a function of the variables $\tilde{\boldsymbol{E}}_{\perp}(\boldsymbol{k},t)=-\dot{\tilde{\boldsymbol{A}}}_{\perp}(\boldsymbol{k},t)$ and $\tilde{\boldsymbol{A}}_{\perp}(\boldsymbol{k},t)$ introduced above for the transverse field:

$$H_{\mathrm{trans}}=\frac{\varepsilon_0}{2}\int \mathrm{d}^3k\;\left[\dot{\tilde{\boldsymbol{A}}}_{\perp}^*(\boldsymbol{k},t)\cdot\dot{\tilde{\boldsymbol{A}}}_{\perp}(\boldsymbol{k},t)+\omega^2\tilde{\boldsymbol{A}}_{\perp}^*(\boldsymbol{k},t)\cdot\tilde{\boldsymbol{A}}_{\perp}(\boldsymbol{k},t)\right] \tag{A-40}$$

Finally, the energy of the global system field + particles can be expressed in the form:

$$H=\frac{1}{2}\sum_a m_a\dot{\boldsymbol{r}}_a^2(t)+V_{\mathrm{Coul}}+H_{\mathrm{trans}} \tag{A-41}$$

where we used the simplified notation $\dot{\boldsymbol{r}}_a(t)=\mathrm{d}\boldsymbol{r}_a(t)/\mathrm{d}t=\boldsymbol{v}_a(t)$. It is the sum of the kinetic energy of the particles, of their Coulomb energy, and of the energy of the transverse field.

A-3-b. Total momentum

Similar computations can be carried out for the total momentum $\boldsymbol{P}$. The field contribution contained in the last term of (A-6b) can be written as:

$$\varepsilon_0 \int \mathrm{d}^3k\ \tilde{\boldsymbol{E}}^*(\boldsymbol{k},t) \times \tilde{\boldsymbol{B}}(\boldsymbol{k},t) = \underbrace{\varepsilon_0 \int \mathrm{d}^3k\ \tilde{\boldsymbol{E}}^*_{\parallel}(\boldsymbol{k},t) \times \tilde{\boldsymbol{B}}(\boldsymbol{k},t)}_{\boldsymbol{P}_{\text{long}}} + \underbrace{\varepsilon_0 \int \mathrm{d}^3k\ \tilde{\boldsymbol{E}}^*_{\perp}(\boldsymbol{k},t) \times \tilde{\boldsymbol{B}}(\boldsymbol{k},t)}_{\boldsymbol{P}_{\text{trans}}} \tag{A-42}$$

where we have separated the contributions to $\boldsymbol{P}$ coming from the longitudinal and transverse components of the electric field[6]. Using (A-18a) and (A-27b), taking into account identity (A-28) and the fact that $\boldsymbol{k} \cdot \tilde{\boldsymbol{A}}_{\perp}(\boldsymbol{k},t) = 0$, we get:

$$\begin{aligned}\boldsymbol{P}_{\text{long}} &= \varepsilon_0 \int \mathrm{d}^3k\ \frac{i\tilde{\rho}^*(\boldsymbol{k},t)}{\varepsilon_0} \frac{\boldsymbol{k}}{k^2} \times \left(i\boldsymbol{k} \times \tilde{\boldsymbol{A}}_{\perp}(\boldsymbol{k},t)\right) \\ &= \int \mathrm{d}^3k\ \tilde{\rho}^*(\boldsymbol{k},t)\tilde{\boldsymbol{A}}_{\perp}(\boldsymbol{k},t)\end{aligned} \tag{A-43}$$

We then have:

$$\begin{aligned}\boldsymbol{P}_{\text{long}} &= \int \mathrm{d}^3r\ \rho(\boldsymbol{r},t)\boldsymbol{A}_{\perp}(\boldsymbol{r},t) \\ &= \sum_a q_a \boldsymbol{A}_{\perp}(\boldsymbol{r}_a,t).\end{aligned} \tag{A-44}$$

As we did above for (A-40), we can rewrite the expression of $\boldsymbol{P}_{\text{trans}}$ as a function of the variables $\tilde{\boldsymbol{E}}_{\perp}(\boldsymbol{k},t) = -\dot{\tilde{\boldsymbol{A}}}_{\perp}(\boldsymbol{k},t)$ and $\tilde{\boldsymbol{A}}_{\perp}(\boldsymbol{k},t)$ of the transverse field:

$$\begin{aligned}\boldsymbol{P}_{\text{trans}} &= -\varepsilon_0 \int \mathrm{d}^3k\ \dot{\tilde{\boldsymbol{A}}}^*_{\perp}(\boldsymbol{k},t) \times \left[i\boldsymbol{k} \times \tilde{\boldsymbol{A}}_{\perp}(\boldsymbol{k},t)\right] \\ &= -i\varepsilon_0 \int \mathrm{d}^3k\ \ \boldsymbol{k}\ \ \left[\dot{\tilde{\boldsymbol{A}}}^*_{\perp}(\boldsymbol{k},t) \cdot \tilde{\boldsymbol{A}}_{\perp}(\boldsymbol{k},t)\right]\end{aligned} \tag{A-45}$$

The momentum of the global system field + particles can be written in the form:

$$\boldsymbol{P} = \sum_a \left[m_a \dot{\boldsymbol{r}}_a(t) + q_a \boldsymbol{A}_{\perp}(\boldsymbol{r}_a,t)\right] + \boldsymbol{P}_{\text{trans}} \tag{A-46}$$

Let us finally introduce the quantity:

$$\boldsymbol{p}_a(t) = m_a \dot{\boldsymbol{r}}_a(t) + q_a \boldsymbol{A}_{\perp}(\boldsymbol{r}_a,t) \tag{A-47}$$

We shall see later that, in the Coulomb gauge electrodynamics, $\boldsymbol{p}_a(t)$ is the conjugate momentum of $\boldsymbol{r}_a(t)$, hence different from the mechanical momentum $m_a\dot{\boldsymbol{r}}_a(t)$. Expressed

[6] The notation $\boldsymbol{P}_{\text{long}}$ should not lead us to believe that $\boldsymbol{P}_{\text{long}}$ is a longitudinal field vector itself: it is actually the vector yielding the longitudinal electric field contribution to the momentum vector; the same comment applies to $\boldsymbol{P}_{\text{trans}}$.

as a function of $\boldsymbol{p}_a(t)$, the total energy (A-41) and total momentum (A-46) are written as:

$$H = \frac{1}{2m_a} \sum_a \left[\boldsymbol{p}_a - q_a \boldsymbol{A}_\perp(\boldsymbol{r}_a, t)\right]^2 + V_{\text{Coul}} + H_{\text{trans}} \tag{A-48}$$

$$\boldsymbol{P} = \sum_a \boldsymbol{p}_a(t) + \boldsymbol{P}_{\text{trans}} \tag{A-49}$$

where H_{trans} and $\boldsymbol{P}_{\text{trans}}$ were introduced in equations (A-38b) and (A-42). We shall see that H actually coincides with the Hamiltonian in the Coulomb gauge of the global system field + particles.

A-3-c. Total angular momentum

Calculations similar to ones just presented, but that will not be detailed here[7], show that the contribution of the longitudinal electric field to the total angular momentum is equal to:

$$\boldsymbol{J}_{\text{long}} = \varepsilon_0 \int \mathrm{d}^3 r \; \boldsymbol{r} \times (\boldsymbol{E}_{\parallel} \times \boldsymbol{B}) = \sum_a q_a \boldsymbol{r}_a \times \boldsymbol{A}_\perp(\boldsymbol{r}_a). \tag{A-50}$$

Adding $\boldsymbol{J}_{\text{long}}$ to the particles' angular momenta, we get, taking (A-47) into account:

$$\sum_a \boldsymbol{r}_a \times m_a \dot{\boldsymbol{r}}_a + \boldsymbol{J}_{\text{long}} = \sum_a \boldsymbol{r}_a \times \boldsymbol{p}_a \tag{A-51}$$

so that we can finally write:

$$\boldsymbol{J} = \sum_a \boldsymbol{r}_a \times \boldsymbol{p}_a + \boldsymbol{J}_{\text{trans}} \tag{A-52}$$

where:

$$\boldsymbol{J}_{\text{trans}} = \varepsilon_0 \int \mathrm{d}^3 r \left[\boldsymbol{r} \times (\boldsymbol{E}_\perp \times \boldsymbol{B})\right] \tag{A-53}$$

B. Describing the transverse field as an ensemble of harmonic oscillators

B-1. Brief review of the one-dimensional harmonic oscillator

The energy of a harmonic oscillator is given by:

$$E = \frac{1}{2} m \dot{x}^2 + \frac{1}{2} m \omega^2 x^2 \tag{B-1}$$

where $\omega/2\pi$ is the oscillation frequency, and $\dot{x}$ the oscillator velocity:

$$\frac{\mathrm{d}}{\mathrm{d}t}\, x = \dot{x} \tag{B-2}$$

[7] These calculations can be found in § 1 of Complement $\mathrm{B_I}$ in [16].

This velocity obeys:

$$\frac{\mathrm{d}}{\mathrm{d}t}\,\dot{x} = -\omega^2 x \tag{B-3}$$

so that the equation of motion of x is:

$$\ddot{x} + \omega^2 x = 0 \tag{B-4}$$

Consequently, the time evolution of $x(t)$ is given by a (real) linear combination of $\cos(\omega t)$ and $\sin(\omega t)$.

The dynamic state of the classical harmonic oscillator is defined at each instant by two real variables $x(t)$ and $\dot{x}(t)$. It is often useful to combine them into a single complex variable $\alpha(t)$ by setting:

$$\alpha(t) = C\left[x(t) + i\,\frac{\dot{x}(t)}{\omega}\right] \tag{B-5}$$

where C is an arbitrary (time-independent) constant. Relations (B-2) and (B-3) show that $\alpha(t)$ obeys the first order differential equation:

$$\dot{\alpha} = C(\dot{x} - i\omega x) = -i\omega C\left(x + i\frac{\dot{x}}{\omega}\right) = -i\omega\alpha \tag{B-6}$$

The time dependence of the new variable $\alpha(t)$ is therefore simply $e^{-i\omega t}$.

One can invert the system formed by equation (B-5) and its complex conjugate yielding α^*, and compute x and $\dot{x}$ as a function of α and α^*. Inserting the expressions thus obtained in equation (B-1) for the energy E, we obtain by a simple calculation[8]:

$$E = \frac{m\omega^2}{4C^2}(\alpha^*\alpha + \alpha\alpha^*) \tag{B-7}$$

The constant C can be chosen so that:

$$\frac{m\omega^2}{4C^2} = \frac{\hbar\omega}{2} \tag{B-8}$$

This leads, after quantization, to the Hamiltonian operator:

$$\hat{H} = \frac{\hbar\omega}{2}(\hat{a}^\dagger\hat{a} + \hat{a}\hat{a}^\dagger) \tag{B-9}$$

which is the Hamiltonian of a harmonic oscillator[9].

B-2. Normal variables for the transverse field

B-2-a. Vibration eigenmodes of the free transverse field

In the reciprocal space, expression (A-40) for the free transverse field energy H_{trans} is a sum of quadratic functions of $\dot{\tilde{\boldsymbol{A}}}_\perp(\boldsymbol{k}, t)$ and $\tilde{\boldsymbol{A}}_\perp(\boldsymbol{k}, t)$. For each value of $\boldsymbol{k}$, we get a harmonic oscillator Hamiltonian. The evolution introduces no coupling between the various spatial Fourier components of the transverse field. We see the advantage of

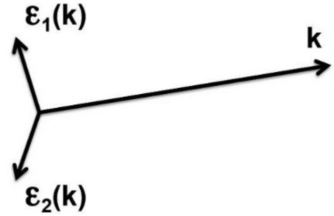

Figure 1: For each vector $\boldsymbol{k}$, the transverse fields can have two polarizations characterized by unit vectors $\boldsymbol{\varepsilon}_1(\boldsymbol{k})$ and $\boldsymbol{\varepsilon}_2(\boldsymbol{k})$ perpendicular both to each other and to $\boldsymbol{k}$.

working in the reciprocal space: it enables us to identify the eigenmodes of the field vibrations, in the absence of sources.

Actually, for each $\boldsymbol{k}$, the transverse field can have two different polarizations [10] characterized by unit vectors $\boldsymbol{\varepsilon}_1(\boldsymbol{k})$ and $\boldsymbol{\varepsilon}_2(\boldsymbol{k})$, both perpendicular to $\boldsymbol{k}$ and to each other, so that we can write for $\tilde{\boldsymbol{A}}_\perp(\boldsymbol{k},t)$, as an example:

$$\tilde{\boldsymbol{A}}_\perp(\boldsymbol{k},t) = \tilde{A}_{\perp\boldsymbol{\varepsilon}_1(\boldsymbol{k})}(\boldsymbol{k},t)\,\boldsymbol{\varepsilon}_1(\boldsymbol{k}) + \tilde{A}_{\perp\boldsymbol{\varepsilon}_2(\boldsymbol{k})}(\boldsymbol{k},t)\,\boldsymbol{\varepsilon}_2(\boldsymbol{k}) = \sum_{\boldsymbol{\varepsilon}_i(\boldsymbol{k})} \tilde{A}_{\perp\boldsymbol{\varepsilon}_i(\boldsymbol{k})}(\boldsymbol{k},t)\,\boldsymbol{\varepsilon}_i(\boldsymbol{k}) \quad \text{(B-10)}$$

with:

$$\tilde{A}_{\perp\boldsymbol{\varepsilon}_i(\boldsymbol{k})}(\boldsymbol{k},t) = \boldsymbol{\varepsilon}_i(\boldsymbol{k})\cdot\tilde{\boldsymbol{A}}_\perp(\boldsymbol{k},t) \quad \text{(B-11)}$$

The set $\{\boldsymbol{k},\boldsymbol{\varepsilon}_i(\boldsymbol{k})\}$ defines what we shall call in this chapter a free field *mode*; they are the eigenmodes of the free field vibration, with a frequency:

$$\omega = ck \quad \text{(B-12)}$$

To simplify the notation, we shall write the last summation in (B-10) in a more compact form:

$$\sum_{\boldsymbol{\varepsilon}_i(\boldsymbol{k})} \tilde{A}_{\perp\boldsymbol{\varepsilon}_i(\boldsymbol{k})}(\boldsymbol{k},t)\,\boldsymbol{\varepsilon}_i(\boldsymbol{k}) \equiv \sum_{\boldsymbol{\varepsilon}} \tilde{A}_{\perp\boldsymbol{\varepsilon}}(\boldsymbol{k},t)\boldsymbol{\varepsilon} \quad \text{(B-13)}$$

Let us rewrite expression (A-40) for $\tilde{H}_{\text{trans}}$ expliciting the components of the fields $\boldsymbol{A}_\perp(\boldsymbol{k},t)$ and $\dot{\boldsymbol{A}}_\perp(\boldsymbol{k},t)$ on the polarization vectors. We get:

$$H_{\text{trans}} = \frac{\varepsilon_0}{2}\int \mathrm{d}^3k \sum_{\boldsymbol{\varepsilon}} \left[\dot{\tilde{A}}^*_{\perp\boldsymbol{\varepsilon}}(\boldsymbol{k},t)\dot{\tilde{A}}_{\perp\boldsymbol{\varepsilon}}(\boldsymbol{k},t) + \omega^2\tilde{A}^*_{\perp\boldsymbol{\varepsilon}}(\boldsymbol{k},t)\tilde{A}_{\perp\boldsymbol{\varepsilon}}(\boldsymbol{k},t)\right] \quad \text{(B-14)}$$

[8] In view of the quantization where α and α^* will be replaced by non-commuting operators $\hat{a}$ and $\hat{a}^\dagger$, we keep the sequence of α and α^* as they appear in the computations.

[9] If $\hat{x}$ and $\hat{p}$ obey the canonical commutation relation $[\hat{x},\hat{p}] = i\hbar$, relation (B-8) for the choice of C also leads to the commutation relation $[\hat{a},\hat{a}^\dagger] = 1$.

[10] We choose real vectors $\boldsymbol{\varepsilon}_1(\boldsymbol{k})$ and $\boldsymbol{\varepsilon}_2(\boldsymbol{k})$ corresponding to linear polarizations, but the choice of these two polarizations is arbitrary, since they can always be rotated by any angle around $\boldsymbol{k}$. It is also possible to perform a more general change of basis with complex vectors defining elliptical polarizations, for instance the right and left circular polarizations $\boldsymbol{\varepsilon}_\pm = (\boldsymbol{\varepsilon}_1 \pm i\boldsymbol{\varepsilon}_2)/\sqrt{2}$. Circular polarizations are useful when discussing electromagnetic spin — see § 3 of Complement B_{XIX}. If complex (orthonormal) polarizations are used, $\boldsymbol{\varepsilon}_i(\boldsymbol{k})$ should be replaced by $\boldsymbol{\varepsilon}_i^*(\boldsymbol{k})$ in the right side hand of relation (B-11).

Note that the components on the two polarizations $\boldsymbol{\varepsilon}$ are truly independent dynamic variables (generalized coordinates and velocities). This is not the case for the Cartesian components $\tilde{A}_{\perp i}(\boldsymbol{k},t)$ and $\dot{\tilde{A}}_{\perp i}(\boldsymbol{k},t)$ (with $i = x, y, z$), because of the transversality condition. For example, the components $\tilde{A}_{\perp i}(\boldsymbol{k},t)$ must obey $\sum_i k_i \tilde{A}_{\perp i} = 0$.

Constraints on the dynamic variables in the reciprocal space

Since the fields are real in real space, we have the condition $\tilde{\boldsymbol{A}}^*_{\perp}(\boldsymbol{k},t) = \tilde{\boldsymbol{A}}_{\perp}(-\boldsymbol{k},t)$. In half the reciprocal space, the variables $\tilde{A}_{\perp\boldsymbol{\varepsilon}}(\boldsymbol{k},t)$ and $\tilde{A}^*_{\perp\boldsymbol{\varepsilon}}(\boldsymbol{k},t)$ can be considered as independent .

B-2-b. Definition of the normal variables, free field case

Let us first assume that we are in the free field case ($\tilde{\boldsymbol{j}}_{\perp} = \boldsymbol{0}$), and we can replace the field $\tilde{\boldsymbol{E}}_{\perp}(\boldsymbol{k},t)$ by $-\dot{\tilde{\boldsymbol{A}}}_{\perp}(\boldsymbol{k},t)$ in equations (A-30a) and (A-30b). As $\omega = kc$, we get two equations exactly similar to those of a harmonic oscillator (B-2) and (B-3), with $\boldsymbol{A}_{\perp}(\boldsymbol{k},t)$ instead of $x(t)$. This analogy suggests introducing, as in (B-5), a new transverse variable:

$$\begin{aligned}\boldsymbol{\alpha}(\boldsymbol{k},t) &= \mathcal{N}(k)\left[\tilde{\boldsymbol{A}}_{\perp}(\boldsymbol{k},t) + \frac{i}{\omega}\dot{\tilde{\boldsymbol{A}}}_{\perp}(\boldsymbol{k},t)\right]\\ &= i\mathcal{N}(k)\left[\frac{\boldsymbol{k}}{k^2}\times\tilde{\boldsymbol{B}}(\boldsymbol{k},t) - \frac{1}{\omega}\tilde{\boldsymbol{E}}_{\perp}(\boldsymbol{k},t)\right]\end{aligned} \quad \text{(B-15)}$$

where $\mathcal{N}(k)$ is a real constant, not yet specified, which can depend on k (its value will be chosen at the beginning of the next chapter). This definition, together with (A-30b), yields the equation of motion for $\boldsymbol{\alpha}(\boldsymbol{k},t)$:

$$\dot{\boldsymbol{\alpha}}(\boldsymbol{k},t) + i\omega\boldsymbol{\alpha}(\boldsymbol{k},t) = 0 \quad \text{(B-16)}$$

As opposed to $\boldsymbol{A}_{\perp}(\boldsymbol{k},t)$ that, according to (A-31), obeys a second order equation, this new variable $\boldsymbol{\alpha}(\boldsymbol{k},t)$ obeys a first order equation. It is a complex variable whose time evolution is proportional to $e^{-i\omega t}$, and not, as is the case for the variable $\boldsymbol{A}_{\perp}(\boldsymbol{k},t)$, to a linear superposition of $e^{-i\omega t}$ and $e^{+i\omega t}$. It will be useful in what follows to consider the complex conjugate of equation (B-15):

$$\begin{aligned}\boldsymbol{\alpha}^*(\boldsymbol{k},t) &= \mathcal{N}(k)\left[\tilde{\boldsymbol{A}}^*_{\perp}(\boldsymbol{k},t) - \frac{i}{\omega}\dot{\tilde{\boldsymbol{A}}}^*_{\perp}(\boldsymbol{k},t)\right]\\ &= \mathcal{N}(k)\left[\tilde{\boldsymbol{A}}_{\perp}(-\boldsymbol{k},t) - \frac{i}{\omega}\dot{\tilde{\boldsymbol{A}}}_{\perp}(-\boldsymbol{k},t)\right]\end{aligned} \quad \text{(B-17)}$$

To go from the first to the second line of (B-17), we used the fact that $\boldsymbol{A}_{\perp}$ is real in the real space, which leads to:

$$\tilde{\boldsymbol{A}}^*_{\perp}(\boldsymbol{k},t) = \tilde{\boldsymbol{A}}_{\perp}(-\boldsymbol{k},t) \quad \text{(B-18)}$$

A similar relation exists for $\dot{\tilde{\boldsymbol{A}}}_{\perp}$. The transverse variables $\boldsymbol{\alpha}(\boldsymbol{k},t)$ and $\boldsymbol{\alpha}^*(\boldsymbol{k},t)$ are called the transverse field *normal variables.* We will see in the next chapter that the quantization process will transform these variables into annihilation and creation operators of photons.

B-2-c. Equation of motion for the normal variables in the presence of sources

In the presence of sources, $\tilde{\boldsymbol{j}}_\perp$ is no longer zero. We can still define the normal variables $\boldsymbol{\alpha}(\boldsymbol{k},t)$ by relations (B-15), but we must now keep the term in $\tilde{\boldsymbol{j}}_\perp(\boldsymbol{k},t)$ on the right-hand side of equation (A-30b). The same transformation that led us from equations (A-30a) and (A-30b) to (B-16) now yields a new equation of motion in the presence of sources:

$$\dot{\boldsymbol{\alpha}}(\boldsymbol{k},t) + i\omega\boldsymbol{\alpha}(\boldsymbol{k},t) = \frac{i\mathcal{N}(k)}{\varepsilon_0\ \omega}\tilde{\boldsymbol{j}}_\perp(\boldsymbol{k},t) \qquad \text{(B-19)}$$

This equation is strictly equivalent to Maxwell's equations for the transverse fields. One can see this by taking the time derivative of equations (B-22a) and (B-22b) given below, and using (B-19) to get the time-dependent evolution equations (A-30a) and (A-30b) of these fields.

Independence of the normal variables

Another interest of the normal variables is that they are independent: there is no relation between $\boldsymbol{\alpha}(\boldsymbol{k},t)$ and $\boldsymbol{\alpha}^*(-\boldsymbol{k},t)$ such as the one that exists between $\tilde{\boldsymbol{A}}_\perp(\boldsymbol{k},t)$ and $\tilde{\boldsymbol{A}}^*_\perp(-\boldsymbol{k},t)$. This is because the real and imaginary parts of $\boldsymbol{\alpha}(\boldsymbol{k},t)$ depend on two independent degrees of freedom, $\tilde{\boldsymbol{A}}_\perp(\boldsymbol{k},t)$ and its time derivative. It is easy to check, by changing the sign of $\boldsymbol{k}$ in (B-15) and by using (B-18) that:

$$\boldsymbol{\alpha}(-\boldsymbol{k},t) = \mathcal{N}(k)\left[\tilde{\boldsymbol{A}}^*_\perp(\boldsymbol{k},t) + \frac{i}{\omega}\dot{\tilde{\boldsymbol{A}}}^*_\perp(\boldsymbol{k},t)\right] \neq \boldsymbol{\alpha}^*(\boldsymbol{k},t) \qquad \text{(B-20)}$$

The knowledge of the $\boldsymbol{\alpha}(\boldsymbol{k},t)$ in the entire reciprocal space does not entail the knowledge of the $\boldsymbol{\alpha}^*(\boldsymbol{k},t)$. Consequently, the integrals over $\boldsymbol{k}$ of the normal variables must be taken over the entire space, and not be limited to half the reciprocal space.

B-2-d. Expression of the physical quantities in terms of the normal variables

We are going to show that all the physical quantities can be expressed in terms of the normal variables.

α. Transverse fields in the reciprocal space

Replacing $\boldsymbol{k}$ by $-\boldsymbol{k}$, we can rewrite equation (B-17) as:

$$\boldsymbol{\alpha}^*(-\boldsymbol{k},t) = \mathcal{N}(k)\left[\tilde{\boldsymbol{A}}_\perp(\boldsymbol{k},t) - \frac{i}{\omega}\dot{\tilde{\boldsymbol{A}}}_\perp(\boldsymbol{k},t)\right] \qquad \text{(B-21)}$$

Using (B-15) and (B-21), we can now express $\tilde{\boldsymbol{A}}_\perp(\boldsymbol{k},t)$ and $\dot{\tilde{\boldsymbol{A}}}_\perp(\boldsymbol{k},t)$ as a function of $\boldsymbol{\alpha}(\boldsymbol{k},t)$ and $\boldsymbol{\alpha}^*(-\boldsymbol{k},t)$. We get:

$$\tilde{\boldsymbol{A}}_\perp(\boldsymbol{k},t) = \frac{1}{2\mathcal{N}(k)}\left[\boldsymbol{\alpha}(\boldsymbol{k},t) + \boldsymbol{\alpha}^*(-\boldsymbol{k},t)\right] \qquad \text{(B-22a)}$$

$$\dot{\tilde{\boldsymbol{A}}}_\perp(\boldsymbol{k},t) = -i\frac{\omega}{2\mathcal{N}(k)}\left[\boldsymbol{\alpha}(\boldsymbol{k},t) - \boldsymbol{\alpha}^*(-\boldsymbol{k},t)\right] \qquad \text{(B-22b)}$$

β. Energy and momentum of the transverse field

We insert relations (B-22a) and (B-22b) for $\tilde{\boldsymbol{A}}_\perp(\boldsymbol{k},t)$ and $\dot{\tilde{A}}_\perp(\boldsymbol{k},t)$ in the expression (A-40) for H_{trans}, using the more compact notation:

$$\boldsymbol{\alpha} = \boldsymbol{\alpha}(\boldsymbol{k},t) \quad ; \quad \boldsymbol{\alpha}_- = \boldsymbol{\alpha}(-\boldsymbol{k},t) \tag{B-23}$$

We get:

$$\begin{aligned} H_{\text{trans}} &= \frac{\varepsilon_0}{2}\int \mathrm{d}^3k\ \frac{\omega^2}{4\mathcal{N}^2(k)}\left[(\boldsymbol{\alpha}^* - \boldsymbol{\alpha}_-)\cdot(\boldsymbol{\alpha} - \boldsymbol{\alpha}_-^*) + (\boldsymbol{\alpha}^* + \boldsymbol{\alpha}_-)\cdot(\boldsymbol{\alpha} + \boldsymbol{\alpha}_-^*)\right] \\ &= \frac{\varepsilon_0}{2}\int \mathrm{d}^3k\ \frac{\omega^2}{4\mathcal{N}^2(k)}\left[2\boldsymbol{\alpha}^*\cdot\boldsymbol{\alpha} + 2\boldsymbol{\alpha}_-\cdot\boldsymbol{\alpha}_-^*\right] \end{aligned} \tag{B-24}$$

(in these equations, we keep the ordered sequence of $\boldsymbol{\alpha}$ and $\boldsymbol{\alpha}^*$ as they appear in the computations, even though $\boldsymbol{\alpha}$ and $\boldsymbol{\alpha}^*$ are commuting numbers; the reason is that similar computations can be carried out in the quantum theory where $\boldsymbol{\alpha}$ and $\boldsymbol{\alpha}^*$ will be replaced by non-commuting operators). A change of variable $\boldsymbol{k} \to -\boldsymbol{k}$ in the integral of the terms in $\boldsymbol{\alpha}_-\cdot\boldsymbol{\alpha}_-^*$ yields an integral of $\boldsymbol{\alpha}\cdot\boldsymbol{\alpha}^*$. We then get:

$$H_{\text{trans}} = \varepsilon_0 \int \mathrm{d}^3k\ \frac{\omega^2}{4\mathcal{N}^2(k)}\left[\boldsymbol{\alpha}^*\cdot\boldsymbol{\alpha} + \boldsymbol{\alpha}\cdot\boldsymbol{\alpha}^*\right] \tag{B-25}$$

Expliciting the components of $\boldsymbol{\alpha}$ and $\boldsymbol{\alpha}^*$ on the two polarization vectors $\boldsymbol{\varepsilon}$ perpendicular to $\boldsymbol{k}$, and using the simplified notation (B-13), we finally get:

$$H_{\text{trans}} = \varepsilon_0 \int \mathrm{d}^3k\ \sum_{\boldsymbol{\varepsilon}} \frac{\omega^2}{4\mathcal{N}^2(k)}\left[\alpha_{\boldsymbol{\varepsilon}}^*(\boldsymbol{k},t)\alpha_{\boldsymbol{\varepsilon}}(\boldsymbol{k},t) + \alpha_{\boldsymbol{\varepsilon}}(\boldsymbol{k},t)\alpha_{\boldsymbol{\varepsilon}}^*(\boldsymbol{k},t)\right] \tag{B-26}$$

This expression looks a lot like a sum of harmonic oscillator Hamiltonians; a suitable choice for the constant $\mathcal{N}$ will be made in the next chapter.

Similar calculations can be carried out for the transverse field momentum $\boldsymbol{P}_{\text{trans}}$[11]. Using equations (A-45), (B-22a) and (B-22b), we get:

$$\boldsymbol{P}_{\text{trans}} = \varepsilon_0 \int \mathrm{d}^3k\ \sum_{\boldsymbol{\varepsilon}} \frac{\omega}{4\mathcal{N}^2(k)}\ \boldsymbol{k}\ \left[\alpha_{\boldsymbol{\varepsilon}}^*(\boldsymbol{k},t)\alpha_{\boldsymbol{\varepsilon}}(\boldsymbol{k},t) + \alpha_{\boldsymbol{\varepsilon}}(\boldsymbol{k},t)\alpha_{\boldsymbol{\varepsilon}}^*(\boldsymbol{k},t)\right] \tag{B-27}$$

γ. Transverse fields in real space

Let us consider first the transverse potential vector $\tilde{\boldsymbol{A}}_\perp(\boldsymbol{k},t)$, whose expression in terms of the normal variables is given by (B-22a). To get its expression in real space, one must, taking (A-11) into account, multiply (B-22a) by $(2\pi)^{-3/2}e^{i\boldsymbol{k}\cdot\boldsymbol{r}}$ and integrate over $\boldsymbol{k}$. Making the change of variable $\boldsymbol{k} \to -\boldsymbol{k}$ in the integral containing $\boldsymbol{\alpha}^*(-\boldsymbol{k},t)$, we finally get:

$$\boldsymbol{A}_\perp(\boldsymbol{r},t) = \frac{1}{(2\pi)^{3/2}}\int \mathrm{d}^3k\ \sum_{\boldsymbol{\varepsilon}} \frac{1}{2\mathcal{N}(k)}\left[\alpha_{\boldsymbol{\varepsilon}}(\boldsymbol{k},t)\,\boldsymbol{\varepsilon}\ e^{i\boldsymbol{k}\cdot\boldsymbol{r}} + \alpha_{\boldsymbol{\varepsilon}}^*(\boldsymbol{k},t)\,\boldsymbol{\varepsilon}^*\ e^{-i\boldsymbol{k}\cdot\boldsymbol{r}}\right] \tag{B-28}$$

[11]The expression of the angular momentum $\boldsymbol{J}_{\text{trans}}$ of the transverse field, in terms of the normal variables, will be computed in Complement $\mathrm{B_{XIX}}$.

This relation (as well as the next two equations) is written in the general case where the polarizations may be complex, elliptical or circular (*cf.* note 10). This is why the term in $\alpha^*_{\boldsymbol{\varepsilon}}(\boldsymbol{k}, t)$ contains a complex conjugate polarization $\boldsymbol{\varepsilon}^*$.

Similar calculations can be carried out for the transverse electric field as well as for the magnetic field. They yield:

$$\boldsymbol{E}_\perp(\boldsymbol{r}, t) = \frac{1}{(2\pi)^{3/2}} \int \mathrm{d}^3k \sum_{\boldsymbol{\varepsilon}} \frac{i\omega}{2\mathcal{N}(k)} \left[\alpha_{\boldsymbol{\varepsilon}}(\boldsymbol{k}, t)\, \boldsymbol{\varepsilon}\, e^{i\boldsymbol{k}\cdot\boldsymbol{r}} - \alpha^*_{\boldsymbol{\varepsilon}}(\boldsymbol{k}, t)\, \boldsymbol{\varepsilon}^*\, e^{-i\boldsymbol{k}\cdot\boldsymbol{r}}\right] \tag{B-29}$$

$$\boldsymbol{B}(\boldsymbol{r}, t) = \frac{1}{(2\pi)^{3/2}} \int \mathrm{d}^3k \sum_{\boldsymbol{\varepsilon}} \frac{ik}{2\mathcal{N}(k)} \left[\alpha_{\boldsymbol{\varepsilon}}(\boldsymbol{k}, t)\, \boldsymbol{\kappa} \times \boldsymbol{\varepsilon}\, e^{i\boldsymbol{k}\cdot\boldsymbol{r}} - \alpha^*_{\boldsymbol{\varepsilon}}(\boldsymbol{k}, t)\, \boldsymbol{\kappa} \times \boldsymbol{\varepsilon}^*\, e^{-i\boldsymbol{k}\cdot\boldsymbol{r}}\right] \tag{B-30}$$

where $\boldsymbol{\kappa}$ has been defined in (A-16) as the unit vector parallel to $\boldsymbol{k}$.

B-3. Discrete modes in a box

So far, we have considered radiation propagating in an infinite space and used continuous Fourier transforms; in relation (A-11), the electric field is expanded on a continuous basis of normalized plane waves $e^{i\boldsymbol{k}\cdot\boldsymbol{r}}/(2\pi)^{3/2}$. It is often useful, however, to use a discrete basis, assuming the radiation to be contained in a box of finite volume, generally defined as a cube of edge length L; this will frequently occur in the next two chapters when dealing with the quantized radiation. The components of each wave vector must obey the boundary conditions in the box[12], and hence take on discrete values:

$$k_{x,y,z} = 2\pi\; n_{x,y,z}/L \tag{B-31}$$

At the end of the computation, nothing prevents us from choosing a very large value of L in order to check that the final result does not depend on L.

Instead of continuous spatial Fourier transforms, one must now introduce discrete Fourier series where each physical quantity is expanded in terms of normalized plane waves $e^{i\mathbf{k}\cdot\mathbf{r}}/L^{3/2}$. The expansion (A-11) of the electric field then becomes:

$$\boldsymbol{E}(\boldsymbol{r}, t) = \frac{1}{L^{3/2}} \sum_{\mathbf{k}} \tilde{\boldsymbol{E}}_{\boldsymbol{k}}(t) e^{i\boldsymbol{k}\cdot\boldsymbol{r}} \tag{B-32}$$

with[13]:

$$\tilde{\boldsymbol{E}}_{\boldsymbol{k}}(t) = \frac{1}{L^{3/2}} \int_{\mathcal{V}} \mathrm{d}^3r\, \boldsymbol{E}(\boldsymbol{r}, t) e^{-i\boldsymbol{k}\cdot\boldsymbol{r}} \tag{B-33}$$

The summation in (B-32) is discrete, and the integral in (B-33) is now limited to the volume $\mathcal{V}$ of the box.

[12] One can choose to impose the field being zero on the walls, but it is generally easier to enforce periodic boundary conditions (B-31), which lead to the same k density of states.

[13] In Appendix I, we used a slightly different definition for the Fourier series, with which the factor $1/L^{3/2}$ would be missing from (B-32), but where (B-33) would contain a factor $1/L^3$. The definition we use here is chosen to directly yield an expansion of $\boldsymbol{E}(\boldsymbol{r}, t)$ on plane waves normalized in the cube.

Note that if the field is zero outside the box, it is obviously possible to use the continuous Fourier transform (A-10) to get the field component $\tilde{\boldsymbol{E}}(\boldsymbol{k},t)$; however, this latter component is different from the discrete component $\tilde{\boldsymbol{E}}_{\boldsymbol{k}}(t)$, because of the coefficients introduced in the definitions. The two components are related by:

$$\tilde{\boldsymbol{E}}_{\boldsymbol{k}}(t) = \left(\frac{2\pi}{L}\right)^{3/2} \tilde{\boldsymbol{E}}(\boldsymbol{k},t) \tag{B-34}$$

The same changes can be made on the Fourier transforms of all the other physical quantities such as the magnetic field, the vector potential, as well as the charge and current densities. The equations in the reciprocal space such as equations (A-17), (A-21), (A-25) and the following, remain valid if we replace the continuous variables $\mathbf{k}$ by discrete variables, since each side of those equations are multiplied by the same factor $(2\pi)^{3/2} L^{-3/2}$. In the case of a zero field outside the box, the $\alpha_{\boldsymbol{\varepsilon}}(\boldsymbol{k},t)$ are also replaced by :

$$\alpha_{\boldsymbol{k},\boldsymbol{\varepsilon}}(t) = \left(\frac{2\pi}{L}\right)^{3/2} \alpha_{\boldsymbol{\varepsilon}}(\boldsymbol{k},t) \tag{B-35}$$

Coming back to ordinary space $(\boldsymbol{r},t)$ via the inverse Fourier transform, we must use relations of the type (B-32) instead of (A-11). Consequently, once we replace in the integral over d^3k the $\tilde{\boldsymbol{E}}_(\boldsymbol{k},t)$ by the $\tilde{\boldsymbol{E}}_{\boldsymbol{k}}(t)$, we must also introduce a multiplicative factor[14]:

$$\int \mathrm{d}^3k \implies \left(\frac{2\pi}{L}\right)^{3/2} \sum_{\mathbf{k}} \tag{B-36}$$

B-4. Generalization of the mode concept

In the absence of sources, the solution of the equation of motion (B-16) for the normal variable $\alpha_{\boldsymbol{\varepsilon}}(\boldsymbol{k},t)$ is very simple, since it is an exponential with an angular frequency $\omega = ck$:

$$\alpha_{\boldsymbol{\varepsilon}}(\boldsymbol{k},t) = \alpha_{\boldsymbol{\varepsilon}}(\boldsymbol{k},0)e^{-i\omega t} \tag{B-37}$$

Inserting (B-37) in the expressions we just obtained for the transverse fields and the other physical quantities, we see that the fields are linear superpositions of progressive plane waves, propagating independently of each other. The free field energy and momentum are the sum of the squared moduli of the various normal variables, each being time-independent and proportional to $|\alpha_{\boldsymbol{\varepsilon}}(\boldsymbol{k},0)|^2$.

The modes $\{\boldsymbol{k},\boldsymbol{\varepsilon}\}$ introduced in this chapter permit expanding the free transverse fields on progressive plane waves. Nevertheless, other expansions on monochromatic waves that are not necessarily plane waves are also possible; they involve other families of modes, as we are now show, coming back to equation (A-31). In the absence of sources, any monochromatic solution of this equation, of the form $\boldsymbol{A}_{\perp}^{(+)}(\boldsymbol{r})e^{-i\omega t}$, necessarily obeys equation:

$$(\Delta + k^2)\boldsymbol{A}_{\perp}^{(+)}(\boldsymbol{r}) = 0 \tag{B-38}$$

[14]The product of the multiplicative factor of (B-34) and that of (B-35) yields the usual factor $(2\pi/L)^3$, obtained directly from (B-31).

(which is simply the Helmholtz equation) with $k = \omega/c$. The plane waves $e^{\pm i\boldsymbol{k}\cdot\boldsymbol{r}}$ are a possible basis of eigenfunctions for this eigenvalue equation, but not the only one. There exists other bases, such as the basis of stationary waves $\cos \boldsymbol{k}\cdot\boldsymbol{r}$ and $\sin \boldsymbol{k}\cdot\boldsymbol{r}$, the basis of multipolar waves (radiation modes with a specific angular momentum, whereas plane waves have a specific linear momentum), or the basis corresponding to Gaussian modes. More generally, any linear combination of plane waves with the same modulus $\boldsymbol{k}$ can become a mode. Whatever basis is chosen, the transverse field energy will be a sum of the squared moduli of normal variables introduced in the expansions of the transverse fields on the eigenfunctions of that basis. The expression of the other physical quantities, however, will only have a simple form in a particular basis. As an example, the momentum of the transverse field is a sum of squared moduli only in the basis of progressive plane waves, whereas the field angular momentum has a simple form only in the basis of multipolar waves.

Note finally that the field can be contained in a cavity with well defined boundary conditions. Finding the eigenfunctions of equation (B-38) obeying these boundary conditions is a way to determine the eigenmodes of this cavity.

To conclude this chapter, we can say that the free radiation field is equivalent to an ensemble of one-dimensional harmonic oscillators associated with the modes $\{\boldsymbol{k}, \boldsymbol{\varepsilon}\}$ labeled by their wave vector and their transverse polarization. Each mode is associated with a field normal variable, similar to the classical variable of the corresponding classical oscillator, and which will become, in the quantization process, the oscillator annihilation operator. The results established in this chapter will be the simple starting point for the radiation quantization explained in the next chapter.